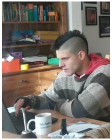
Bindings as Bounded Natural Functors

Jasmin Blanchette, Lorenzo Gheri, Andrei Popescu, Dmitriy Traytel



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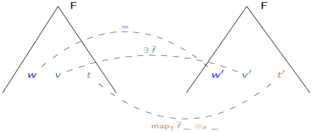

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https://www.youtube.com/watch?v=gkq2ONqQ2MI

Our Contribution

Modular framework for datatypes with bindings

- Complex variable binders
- Infinitary syntax too (including coinductive datatypes)

Formalized in the Isabelle/HOL proof assistant

It is being implemented as a definitional package

What is a Binder?

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Several very expressive syntactic formats:

- CαMl [Pottier 2006]
- Ott [Sewell et al. 2010]
- Unbound [Weirich et al. 2011]
- Isabelle Nominal2 [Urban and Kaliszyk 2012]
- Needle&Knot [Keuchel et al. 2016]

What is a Binder?

Binder = Mechanism for combining any variables with any terms.

$$\lambda v.\, t$$

$$\text{let } v = t_1 \text{ in } t_2$$

$$\text{let rec } v_1 = t_1 \text{ and } \ldots \text{ and } v_k = t_k \text{ in } t$$

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Proposal: Binder = Operator on sets $F : \mathrm{Set}^m \times \mathrm{Set}^n \to \mathrm{Set}$
plus binding dispatcher relation $\theta \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$.

Think: $F\,(V_1, \ldots, V_m, T_1, \ldots, T_n)$ combines variables $v_i \in V_i$ and terms $t_j \in T_j$ such that $v_i \in V_i$ binds in $t_j \in T_j$ if $(i, j) \in \theta$.

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$\lambda v.\, t$	$m = n = 1$ $\mathrm{F}(V, T) = V \times T$ $\theta = \{(1, 1)\}$
let $v = t_1$ in t_2	$m = 1,\ n = 2$ $\mathrm{F}(V, T_1, T_2) = V \times T_1 \times T_2$ $\theta = \{(1, 2)\}$
let rec $v_1 = t_1$ and $\ldots$ and $v_k = t_k$ in t	$m = n = 1$ $\mathrm{F}(V, T) = \mathrm{List}(V \times T) \times T$ $\theta = \{(1, 1)\}$

Structure of Binders

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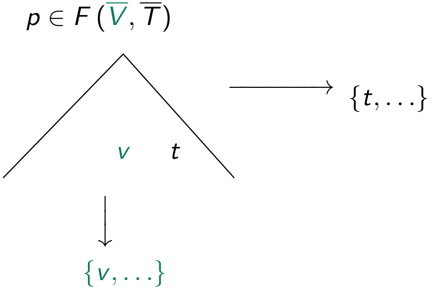
F "Natural" (Container-like)



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Finitary?

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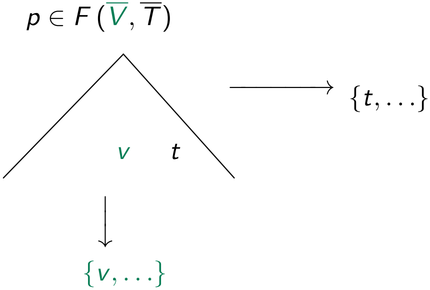

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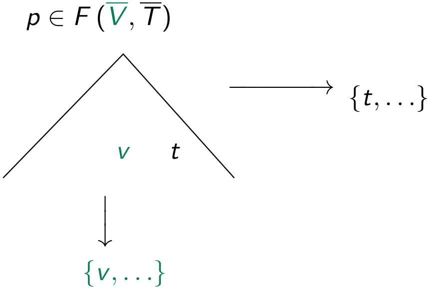

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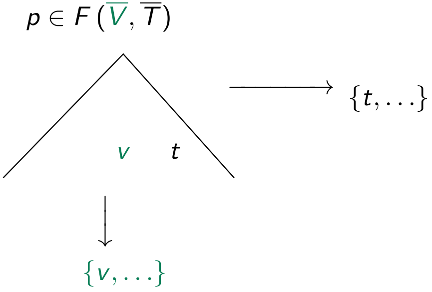

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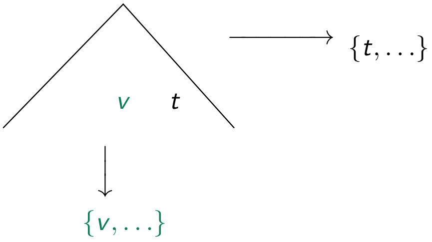
	Functor?

	$\mathrm{F}\,(V, T) =$	let rec $v = t_1$ and $v = t_2$ in t
let rec $(v = t)^*$ in t	$\mathrm{List}\,(V \times T) \times T$	
	$\theta = \{(1, 1)\}$	$[(v, t_1), (v, t_2)] \in \mathrm{List}\,(V \times T)$

$p \in F\,(\overline{V}, \overline{T})$



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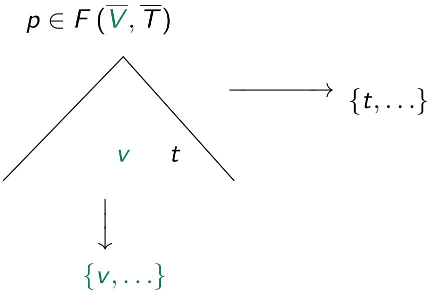

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F	"Natural" (Container-like)
	Functor on (binding) variable arguments only w.r.t. injections

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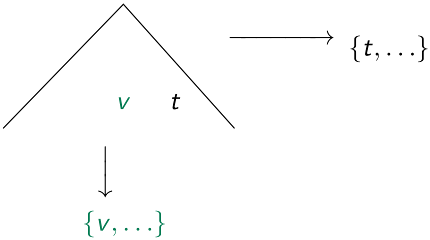

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$w\,(v).\ p$

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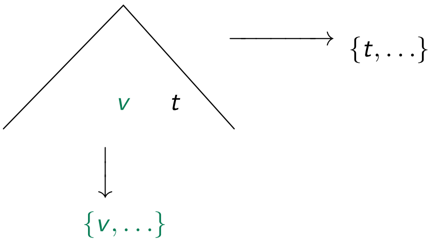

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$w\,(v).\ p$	$p = m = n = 1$ $\mathsf{F}\,(W, V, T) = W \times V \times T$ $\theta = \{(1, 1)\}$

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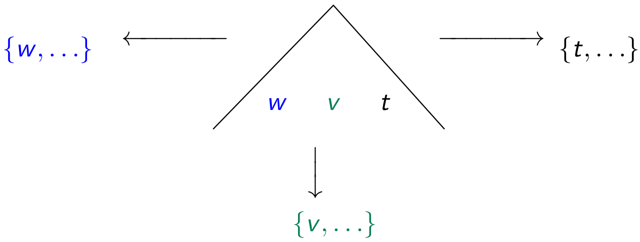
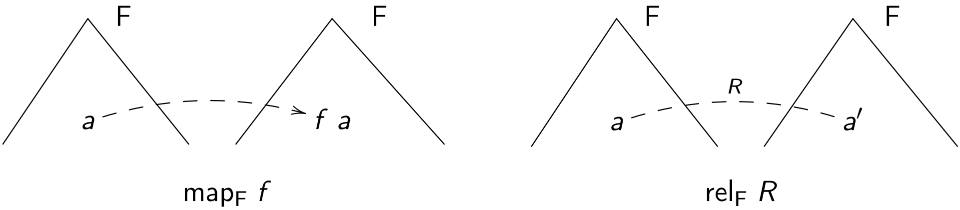


More About Container-Like Functors

	F	:	$\mathsf{Set} \to \mathsf{Set}$
Functor	map_F	:	$\prod_{A,B \in \mathsf{Set}} (A \to B) \to \mathsf{F}\,(A) \to \mathsf{F}\,(B)$
Relator	rel_F	:	$\prod_{A,B \in \mathsf{Set}} \mathcal{P}\,(A \times B) \to \mathcal{P}(\mathsf{F}\,(A) \times \mathsf{F}\,(B))$



Container types [Hoogendijk and de Moor 2000]
Containers [Abbott et al. 2005]
Bounded Natural Functors (BNFs) [Traytel et al. 2012]

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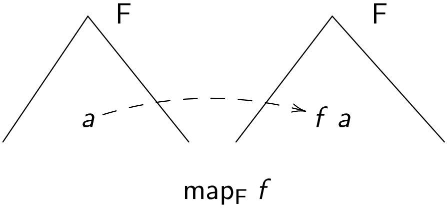

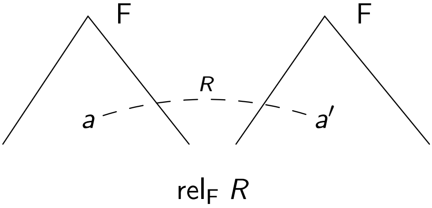

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Structure of Binders (Summary)

Proposal: Binder = Operator on sets $F : \mathsf{Set}^p \times \mathsf{Set}^m \times \mathsf{Set}^n \to \mathsf{Set}$
that is a Map-Restricted Bounded Natural Functor (MRBNF):

w.r.t. arbitrary functions on the p free-variable arguments

w.r.t. injections on the m binding-variable arguments

w.r.t. arbitrary functions on the n "term" arguments

plus binding dispatcher relation $\theta \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$.

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plus binding dispatcher relation $\theta \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$.

Term-agnostic: Binds any hypothetical terms.

Constructing Terms from Binders

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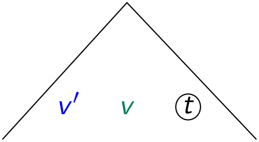
$$F : \mathsf{Set}^p \times \mathsf{Set}^m \times \mathsf{Set} \to \mathsf{Set}$$

Assume $p = m$.

$$\overline{\mathsf{T}\,(\overline{V})} = \mu\overline{A}.\,\mathsf{F}(\overline{V}, \overline{V}, \overline{A})$$



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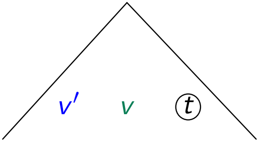

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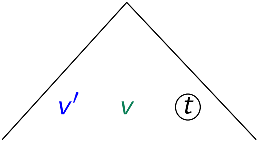
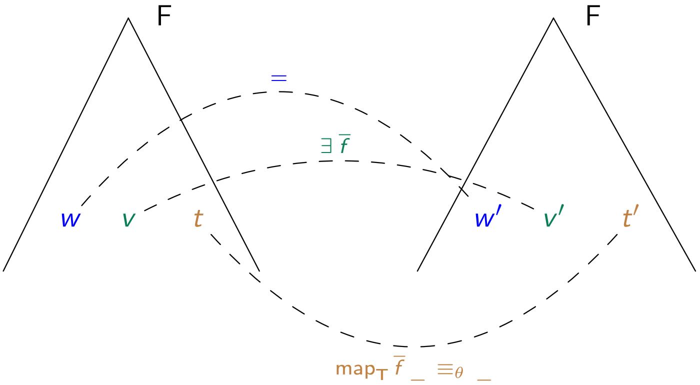


Alpha-quotiented terms: $\overline{\mathbf{T}\,(\overline{V})} = \overline{\mathsf{T}\,(\overline{V})\,/\!\equiv_\theta}$

Inductive Definition of Alpha-Equivalence



Equality on the top free variables
Possible bijective renamings of top binding variables
Recursive call factoring in the renamings

Abstract Characterization of Alpha-Quotinented Terms?

$$\begin{array}{lcll} \mathsf{T}\,(\overline{V}) & = & \mu A.\,\mathsf{F}(\overline{V},\overline{V},A) & \text{OK} \\ \mathbf{T}\,(\overline{V}) & = & \mathsf{T}\,(\overline{V})\,/\equiv_\theta & \text{too low-level} \end{array}$$

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Operators on $\mathbf{T}$:

- ctor : $\mathsf{F}\,(\overline{V}, \overline{V}, \mathbf{T}\,(\overline{V})) \to \mathbf{T}\,(\overline{V})$ non-injective constructor
- $\mathsf{FVars}_i : \mathbf{T}\,(\overline{V}) \to V_i$
- $\mathsf{map}_{\mathbf{T}}$ functorial action on $\mathbf{T}$ w.r.t. bijections

Theorem: $(\mathbf{T}, \overline{\mathsf{FVars}}, \mathsf{map}_{\mathbf{T}}, \mathsf{ctor})$ is the initial object in a category of models $\mathcal{U} = (\mathsf{U}, \overline{\mathsf{UFVars}}, \mathsf{Umap}, \mathsf{Uctor})$ satisfying:

- Umap functorial on bijections
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$$\Downarrow$$

Recursor generalizing the state-of-the-art nominal recursors (Norrish 2004, Pitts 2006, Urban and Berghofer 2006, GP 2017)

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Closure Properties for MRBNFs

Include standard type constructors: sums, products, ...

Include nonfree type constructors: fin. sets, bags, prob. distrib., ...

Closed under standard least/greatest fixpoints: lists, trees, ...

Closed under linearization

Closed under binding-aware least/greatest fixpoints (modulo binding dispatchers)
"Side effects": Binding-aware (co)recursors and (co)induction principles (obeying Barendregt's variable convention)

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$$\Downarrow$$

Modular and flexible specification framework for binding datatypes

- Plug and play
- Complex binders made easy

Example: POPLmark Syntax Fragment

Type-variable α, term-variables x, labels l

$$
\begin{array}{llll}
\text{Types} & \sigma & ::= & \alpha \mid \ldots \\
\text{Patterns} & p & ::= & x : \sigma \mid \{ l_i = p_i \ ^{i \in 1 \ldots n} \} \\
\text{Terms} & t & ::= & x \mid \Lambda\alpha.\ t \mid \text{let } p = t_1 \text{ in } t_2
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Assumptions: Term-variables are pairwise distinct in any pattern.
In terms, term-variables coming from patterns and type-variables near Λ's are binding.

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& = & \mu_\theta T.\ F\,(A, X, A, X, T)
\end{array}
$$

where:

$$F\,(A', X', A, X, T) = X' + A \times T + \text{Pattern}\,(A', X) \times T^2$$

$$\theta = \{(1,1), (2,1)\}$$

Related Work: 1999

A lot of work on categorical generalizations of the "nameless", De Bruijn representation, pioneered by:

Fiore et al. (LICS'99)

Hofmann (LICS'99)

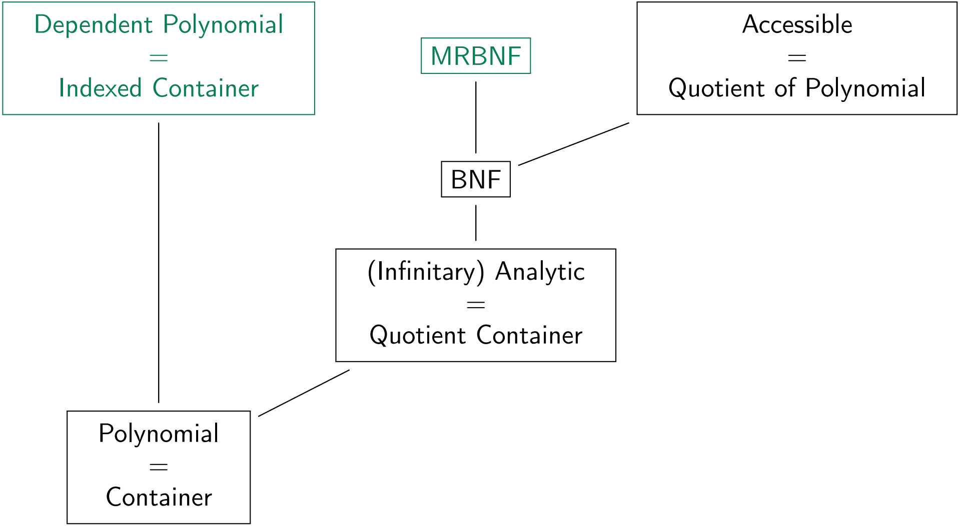
Bird and Paterson (J. Func. Prog. '99)

Altenkirch and Reus (CSL'99)

By contrast, our work is within the "nameful" paradigm, generalizing Nominal Logic (Gabbay and Pitts (LICS'99))

(Higher Order Abstract Syntax (HOAS) – the third main paradigm)

Related Work: Relevant Classes of Functors



Related Work: Binders in the Isabelle Ecosystem

Isabelle Nominal2 [Urban and Kaliszyk 2012]

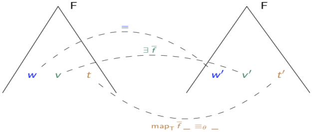
- Good user support
- Complex binders via syntactic format

Our improvements (once the implementation is ready):

- Expressiveness
- Modularity
- Better integration with Isabelle's standard datatypes (which are based on BNFs)

Bindings as Bounded Natural Functors

Jasmin Blanchette, Lorenzo Gheri, Andrei Popescu, Dmitriy Traytel



Vrije Universiteit Amsterdam Middlesex University London ETH Zürich

https://www.youtube.com/watch?v=gkq2ONqQ2MI

Reserve Slides

The Linearization Operator @

$$\mathsf{List}\,(A)^{@A} = \{xs \in \mathsf{List}\,(A) \mid \forall i, j.\ i \neq j \longrightarrow xs_i \neq xs_j\}$$

The Linearization Operator $^{@}$

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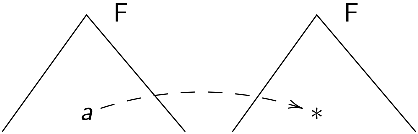
Let F : Set → Set be a BNF.

$$\begin{array}{rcl} !_A & : & A \to \mathsf{Unit} = \{*\} \\ \mathsf{shape} = \mathsf{map_F}\,!_A & : & \mathsf{F}\,(A) \to \mathsf{F}\,(\mathsf{Unit}) \end{array}$$



How about: $p \in F\,(A)$ linear
... if $\forall q.\ \mathsf{shape}\ q = \mathsf{shape}\ p \longrightarrow |\mathsf{set_F}\ q| \leq |\mathsf{set_F}\ p|$

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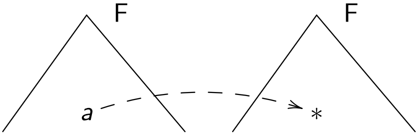

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Works for finitary functors.

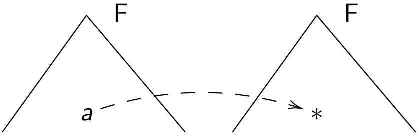
Fails in general: For $\mathsf{F} = \mathsf{Stream}$, $[0, 0, 1, 2, 3, \ldots] \in \mathsf{F}\,(\mathbb{N})$ linear.

The Linearization Operator $^@$

$\mathsf{List}\,(A)^{@A} = \{xs \in \mathsf{List}\,(A) \mid \forall i, j.\ i \neq j \longrightarrow xs_i \neq xs_j\}$

Let $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$ be a BNF.

$$\begin{aligned} !_A &: A \to \mathsf{Unit} = \{*\} \\ \mathsf{shape} = \mathsf{map}_\mathsf{F}\, !_A &: \mathsf{F}\,(A) \to \mathsf{F}\,(\mathsf{Unit}) \end{aligned}$$



Better: $p \in F\,(A)$ linear
... if $\forall q.\ \mathsf{shape}\ q = \mathsf{shape}\ p \longrightarrow \exists f : A \to A.\ \mathsf{map}_\mathsf{F}\ f\ p = q$

Works in general.

Gives us back a sub-functor, $\mathsf{F}^@$, of F's restriction to bijections.

Modularity

Let $F : \mathsf{Set}^m \times \mathsf{Set}^m \times \mathsf{Set} \to \mathsf{Set}$ be an MRBNF.

$$\mathbf{T}\,(\overline{V}) \;=\; \mu_\theta A.\, \mathsf{F}\,(\overline{V}, \overline{V}, A)$$

Is $\mathbf{T}$ also an MRBNF?

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$\mathrm{set}_i^{\mathbf{T}} := \mathrm{FVars}_i$

On bijections, $\mathrm{map}_{\mathbf{T}}$ lifted from $\mathrm{map}_{\mathsf{T}}$.
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But want $\mathsf{map}_{\mathbf{T}}$ non-bijective/injective functions too!

$$\Downarrow$$

$\mathsf{map}_{\mathbf{T}} :=$ the capture-avoiding substitution

Problem: Substitution only behaves well on functions of small support.

So we have F functorial w.r.t. (arbitrary) functions
implies $\mathbf{T}$ functorial only w.r.t. small-support functions.

Solution

Proposal: Binder = Operator on sets $F : \mathsf{Set}^p \times \mathsf{Set}^m \times \mathsf{Set}^n \to \mathsf{Set}$
that is a Map-Restricted Bounded Natural Functor (MRBNF):

- w.r.t. arbitrary functions on the p free-variable arguments
- w.r.t. injections on the m binding-variable arguments
- w.r.t. arbitrary functions on the n "term" arguments

plus binding dispatcher relation $\theta \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$.

$$\Downarrow$$

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Structure of Binders: More Details

Proposal: Binder = Operator on sets $F : \mathsf{Set}^m \times \mathsf{Set}^n \to \mathsf{Set}$
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Can assume more about F?

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Can assume more about F? How about Finitary Natural Functor (FNF)?

Functor: for $f_i : V_i \to V_i'$, $g_j : T_j \to T_j'$

$\mathsf{map_F}\ \overline{f}\ \overline{g} : F\,(\overline{V}, \overline{T}) \to F\,(\overline{V'}, \overline{T'})$

Natural (container-like): there exist the natural transformations

$\mathsf{set}^i_F : F\,(\overline{V}, \overline{T}) \to \mathcal{P}\,(V_i) \quad \mathsf{set}^{m+j}_F : F\,(\overline{V}, \overline{T}) \to \mathcal{P}\,(T_j)$

Finitary: $\mathsf{set}^i_F\,(\ldots)$, $\mathsf{set}^{m+j}_F\,(\ldots)$ finite

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Bounded: For some cardinal $\mathsf{bd_F}$

$$\mathsf{set}_F^i\,(\ldots) \leq \mathsf{bd_F} \quad \mathsf{set}_F^{m+j}\,(\ldots) \leq \mathsf{bd_F}$$

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Problem 1:

let rec $v_1 = t_1$ and $\ldots$ and $v_k = t_k$ in t

$m = n = 1$
$\mathsf{F}\,(V, T) = \mathsf{List}\,(V \times T) \times T$
$\theta = \{(1, 1)\}$

Want to disallow repetitions, as in "let rec $v = t_1$ and $v = t_2$ in t" yet $[(v, t_1), (v, t_2)] \in \mathsf{List}\,(V \times T)$

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Problem 2:

$w\left(v\right).\ p$

Free variables are currently completely ignored.

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