3. Coinductive Predicates

Example of Coinductive Definition

Informal example 3 (the $subll$ predicate) revisited

Given a set A, let LazyList(A) be the set of "lazy lists" (finite or infinite lists) with elements in A – they have the form $[a_1, a_2, \ldots, a_n]$ or $[a_1, a_2, \ldots]$. We write $a \# as$ for the lazy list obtained by consing a to as.

We wish to define the sublist relation, $subll$, on lazy lists.

The relation $subl$ on (finite) lists is defined inductively by the rules:

$$\frac{\cdot}{subl\ []\ as}\ \text{(Nil)} \qquad \frac{subl\ as\ as'}{subl\ as\ (a\#as')}\ \text{(ConsR)}$$

$$\frac{subl\ as\ as'}{subl\ (a\#as)\ (a\#as')}\ \text{(Cons)}$$

The inductive interpretation means: smallest relation closed under the rules (Nil), (ConsR) and (Cons).

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Should we rather go for the greatest relation closed under these rules?
No! This would give us the total relation $\lambda as, as'.\ \top$.
Let's take it easy, starting with selecting the properties that we want...

Desired properties for the predicate $subll$

Say $A = \mathbb{N}$.

For finite lists, $subll$ should behave just like $subl$, e.g.,

- $subll\ [1, 3, 4]\ [1, 2, 3, 4]$
- $subll\ [1, 2]\ [1, 2, 3, 4]$
- $subll\ [1, 3]\ [1, 2, 3, 4]$

Also, e.g.,

- $subll$ zeros zeros, in fact $subll\ as\ as$ for any as
- $subll\ [0, 2, 4, 6, \ldots]\ [0, 1, 2, 3, \ldots]$

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- $subll\ [0,2,4,6,\ldots]\ [0,1,2,3,\ldots]$

$subll\ as\ as'$ should hold if and only if:
assuming as' has the form $[a'_i]_{i<length\ as'}$ (with $length\ as' \in \mathbb{N} \cup \{\infty\}$)
there exists $[j_p]_{p<length\ as}$ such that $\forall p.\ p+1 < length\ as \longrightarrow j_p < j_{p+1}$
and $as = [a'_{j_p}]_{p<length\ as}$.

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Desired properties for the predicate $sublll$

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$$\frac{\begin{array}{c} sublll\ bs\ bs' \qquad \forall as.\ bs = [] \wedge bs' = as \longrightarrow P \\ \forall as, as', a.\ bs = as \wedge bs' = a\#as' \wedge sublll\ as\ as' \longrightarrow P \\ \forall as, as', a.\ bs = a\#as \wedge bs' = a\#as' \wedge sublll\ as\ as' \longrightarrow P \end{array}}{P}\ \text{(Cases)}$$

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or, equivalently...

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How about induction?

$$\frac{\begin{array}{c} subll\ bs\ bs' \qquad\qquad \forall as.\ P\ []\ as \\ \forall as, as', a.\ subll\ as\ as' \wedge P\ as\ as' \longrightarrow P\ as\ (a\#as') \\ \forall as, as', a.\ subll\ as\ as' \wedge P\ as\ as' \longrightarrow P\ (a\#as)\ (a\#as') \end{array}}{P\ bs\ bs'}\ \text{(Induct)}$$

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It would allow us to prove, e.g., $sublll\ as\ as'$ implies as is finite.

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It would allow us to prove, e.g., $sublL\ as\ as'$ implies as is finite.

This would imply, e.g.,

- $\neg\ sublL$ zeros zeros
- $\neg\ sublL\ [0, 2, 4, 6, \ldots]\ [0, 1, 2, 3, \ldots]$

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How can we prove $sublll\ [0, 2, 4, 6, \ldots]\ [0, 1, 2, 3, 4, 5, \ldots]$?

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Desired properties for the predicate $sublll$

This leads to the coinduction rule for $subll$:

$$\frac{P\ cs\ cs' \qquad \forall\, bs, bs'.\ P\ bs\ bs' \longrightarrow \begin{array}{l}(\exists as.\ bs = [\,] \wedge bs' = as) \vee \\ (\exists as, a, as'.\ bs = as \wedge bs' = a\#as' \wedge P\ as\ as') \vee \\ (\exists a, as, as'.\ bs = a\#as \wedge bs' = a\#as' \wedge P\ as\ as')\end{array}}{subll\ cs\ cs'}\ \text{(Coinduct)}$$

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Coinduction says: If a relation P is consistent with the introduction rules (Nil), (ConsR) and (Cons), then $P\ cs\ cs'$ implies $subll\ cs\ cs'$ (for every cs, cs'), i.e., $P \leq subll$.

Desired properties for the predicate $sublI$

This leads to the coinduction rule for $sublI$:

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Terminology: <u>consistent</u> with some rules = “closed backwards” under these rules

Coinduction says: If a relation P is consistent with the introduction rules (Nil), (ConsR) and (Cons), then $P\ cs\ cs'$ implies $sublI\ cs\ cs'$ (for every cs, cs'), i.e., $P \leq sublI$.
In other words, $sublI$ is the greatest (largest) relation that is consistent with the introduction rules.

Desired properties for the predicate $subll$

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Recipe for making sense of coinductive specifications

The relation $subll : \mathsf{LazyList}(A) \to \mathsf{LazyList}(A) \to \mathsf{Bool}$
specified coinductively by the rules:

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F is monotonic, so J_F exists by Knaster-Tarski.

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$subll$ is also the greatest (post-)fixpoint of
$G = \lambda P.\ F\ (subll \vee P) = \lambda P.\ F\ (\lambda as, as'.\ subll\ as\ as' \vee P\ as\ as')$.

Induction versus Coinduction

The semantic foundations for induction and coinduction are perfectly dual – via Knaster-Tarski:

- induction: smallest/least pre-fixpoint
- coinduction: largest/greatest post-fipoint

But they have quite different intuitions:

- induction – whatever can be proved using a finite number of rule applications
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Can we make this intuition precise?

Fix a set A. A rule over A is a pair $r = (H, a)$, $H \subseteq A$ is a finite set and $a \in A$.

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We fix $\mathcal{R}$, a set of rules over A.
We define/specify $\mathsf{I}_{\mathcal{R}}$ inductively by the rules in $\mathcal{R}$, namely:

$$\frac{(H, a) \in \mathcal{R} \qquad\qquad \forall b \in H.\ \mathsf{I}_{\mathcal{R}}\ b}{\mathsf{I}_{\mathcal{R}}\ a}$$

We define/specify $\mathsf{J}_{\mathcal{R}}$ coinductively by the same rules, namely:

$$\frac{(H, a) \in \mathcal{R} \qquad\qquad \forall b \in H.\ \mathsf{J}_{\mathcal{R}}\ b}{\mathsf{J}_{\mathcal{R}}\ a}$$

According to our semantic recipe, the above mean:

We define $F : (A \to \mathsf{Bool}) \to (A \to \mathsf{Bool})$, the operator associated to $\mathcal{R}$, by applying the rules to its input predicate (like we did before in our examples):

$$F\ P\ =\ \lambda a.\ \exists H.\ (H, a) \in \mathcal{R}\ \wedge (\forall a \in H.\ P\ a)$$

F is monotonic, so I_F and J_F exist by Knaster-Tarski.

We define

- $\mathsf{I}_{\mathcal{R}} = \mathsf{I}_F$
- $\mathsf{J}_{\mathcal{R}} = \mathsf{J}_F$

An $\mathcal{R}$-proof tree π is a (possibly infinite) tree whose nodes are labeled with elements of A and such that successor nodes correspond to rules; more precisely, if a node N is labeled with a and its successor nodes $N_1, \ldots, N_k$ are labelled with $a_1, \ldots, a_k$, then $(\{a_1 \ldots a_k\}, a) \in \mathcal{R}$.

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Rule-based definitions

An *$\mathcal{R}$-proof tree* π is a (possibly infinite) tree whose nodes are labeled with elements of A and such that successor nodes correspond to rules; more precisely, if a node N is labeled with a and its successor nodes $N_1, \dots, N_k$ are labelled with $a_1, \dots, a_k$, then $(\{a_1 \dots a_k\}, a) \in \mathcal{R}$.

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Assuming $(\{c, b\}, a)$, $(\{d, b\}, b)$, $(\varnothing, c)$, $(\varnothing, d)$, $(\varnothing, b) \in \mathcal{R}$, here's an example of a finite $\mathcal{R}$-proof tree:

$$\dfrac{\dfrac{.}{\mathsf{c}} \qquad \dfrac{\dfrac{.}{\mathsf{d}} \quad \dfrac{.}{\mathsf{b}}}{\mathsf{b}}}{\mathsf{a}}$$

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Assuming $(\{c, b\}, a)$, $(\{d, b\}, b)$, $(\varnothing, c)$, $(\varnothing, d)$, $(\varnothing, b) \in \mathcal{R}$, and here's an example of an infinite $\mathcal{R}$-proof tree:

$$\dfrac{\dfrac{\cdot}{c} \qquad \dfrac{\dfrac{\cdot}{d} \qquad \dfrac{\dfrac{\cdot}{d} \qquad \dfrac{\dfrac{\cdot}{d} \qquad \dfrac{\dfrac{\cdot}{d} \qquad \vdots}{b}}{b}}{b}}{b}}{a}$$

Characterization Theorem

IMO, crucial for the understanding of coinduction

Theorem. For all $a \in A$:
(1) $\mathsf{I}_{\mathcal{R}}\ a$ holds iff there exists a finite $\mathcal{R}$-proof tree that proves a.
(2) $\mathsf{J}_{\mathcal{R}}\ a$ holds iff there exists a (possibly infinite) $\mathcal{R}$-proof tree that proves a.

Note: It's not really about finite versus infinite – that is just a coincidence from using finitely branching rule systems.

Remove the restriction that rules (A, a) have the set of hypotheses A finite.
Say a tree is well-founded is it has no infinite paths.

More General Version. For all $a \in A$:
(1) $\mathsf{I}_{\mathcal{R}}\ a$ holds iff there exists a well-founded $\mathcal{R}$-proof tree that proves a.
(2) $\mathsf{J}_{\mathcal{R}}\ a$ holds iff there exists a (possibly non-well-founded) $\mathcal{R}$-proof tree that proves a.